

# Spectrum of the Signless 1-Laplacian and the Dual Cheeger Constant on Graphs

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July 5, 2016

## Abstract

Parallel to the signless Laplacian spectral theory, we introduce and develop the non-linear spectral theory of signless 1-Laplacian on graphs. Again, the first eigenvalue  $\mu_1^+$  of the signless 1-Laplacian precisely characterizes the bipartiteness of a graph and naturally connects to the maxcut problem. However, the dual Cheeger constant  $h^+$ , which has only some upper and lower bounds in the Laplacian spectral theory, is proved to be  $1 - \mu_1^+$ . The structure of the eigenvectors and the graphic feature of eigenvalues are also studied. The Courant nodal domain theorem for graphs is extended to the signless 1-Laplacian. A set-pair version of the Lovász extension, which aims at the equivalence between discrete combination optimization and continuous function optimization, is established to recover the relationship  $h^+ = 1 - \mu_1^+$ . A local analysis of the associated functional yields an inverse power method to determine  $h^+$  and then produces an efficient implementation of the recursive spectral cut algorithm for the maxcut problem.

**Keywords:** signless graph 1-Laplacian; dual Cheeger constant; maxcut problem; Lovász extension; inverse power method; nodal domain; spectral graph theory

**AMS subject classifications:** 05C85; 90C27; 58C40; 35P30; 05C50

## 1 Introduction

As one of typical objects in the linear spectral theory, the classical (normalized) graph Laplacian is directly related to two basic properties of graph  $G$  — connectedness and bipartiteness — through the second eigenvalue  $\lambda_2$  and the largest eigenvalue  $\lambda_n$ , respectively, when all eigenvalues are arranged in an increasing order:  $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2$  with  $n$  being the size of  $G$ . Namely, there holds explicitly:

$$\lambda_2 > 0 \Leftrightarrow G \text{ is connected};$$

and then for a connected graph  $G$ ,

$$\lambda_n = 2 \Leftrightarrow G \text{ is bipartite}.$$

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To further quantify the global connectedness, one recourse to the Cheeger constant  $h$ , which was first introduced in geometry by Cheeger [1] and later extended into the discrete setting to find a balanced graph cut, i.e., the Cheeger cut problem [2]. The second eigenvalue is bounded from up and below by the Cheeger constant as stated in the Cheeger inequality [3]. Amazingly, such inequality shrinks to an equality once replacing the graph Laplacian by the graph 1-Laplacian [4, 5]. That is, the second eigenvalue of 1-Laplacian equals to the Cheeger constant, and more importantly the corresponding eigenvector provides an exact Cheeger cut. This equivalence between the Cheeger cut problem and the graph 1-Laplacian based continuous optimization paves a way in solving the Cheeger cut problem [4, 6, 7]. Alternatively, there is another effective approach, the Lovász extension [8], to realize the gapless transformation from the discrete combination optimization to the continuous function optimization providing the discrete objective function can be expressed as a submodular set function. This is exactly the case for the Cheeger cut problem [9].

A similar story for the global bipartiteness and the largest eigenvalue  $\lambda_n$  happens. In fact, the bipartiteness can be directly determined by the maxcut problem as follows

$$h_{\max}(G) = 1 \Leftrightarrow G \text{ is bipartite,}$$

where the maxcut ratio is defined as

$$h_{\max}(G) := \max_{V_1 \subset V} \frac{2|E(V_1, V_1^c)|}{\text{vol}(V)}, \quad (1.1)$$

and  $|E(V_1, V_1^c)|$  sums the weight of edges that cross  $V_1$  and its complement set  $V_1^c$ , while  $\text{vol}(V)$  sums the degree of vertices in  $V$ . However, solving analytically the maxcut problem is combinatorially NP-hard [10] and some approximate solutions or relaxed strategies are then introduced, like the semidefinite programming (SDP) [11, 12] and the advanced scatter search [13] approaches. Recently, a recursive spectral cut (RSC) algorithm, the first algorithm with a numerical solution the cost of which is strictly larger than 1/2 that is not based on SDP, was proposed based on the bipartiteness ratio [14]

$$h^+(G) = \max_{V_1 \cap V_2 = \emptyset, V_1 \cup V_2 \neq \emptyset} \frac{2|E(V_1, V_2)|}{\text{vol}(V_1 \cup V_2)} \quad (1.2)$$

to approach  $h_{\max}(G)$ . Obviously,  $0 < h_{\max}(G) \leq h^+(G) \leq 1$ . This bipartiteness ratio  $h^+(G)$  is also called the dual Cheeger constant with a mathematical motivation to study the spectral gap  $2 - \lambda_n$  and it finally comes up with the so-called dual Cheeger inequality [15]

$$2h^+(G) \leq \lambda_n \leq 1 + \sqrt{1 - (1 - h^+(G))^2}, \quad (1.3)$$

which was also mentioned in [14]. Compared to the story for the Cheeger constant  $h$ , we naturally ask: *Is there an analog of nonlinear spectral graph theory for the dual Cheeger constant  $h^+$  which would reduce the inequality (1.3) to an equality?* The existing theoretical approaches cannot give an answer. On one hand, the graph 1-Laplacian based theory deals with only the connectedness and the Cheeger constant. On the other hand, the original Lovász extension fails because the numerator  $|E(V_1, V_2)|$  cannot be a submodular set function for disjoint sets  $V_1$  and  $V_2$ . Set against such a background, this work will try to give a positive answer.

In order to answer the above question, we introduce the signless graph 1-Laplacian and establish its spectral theory, with which the dual Cheeger inequality (1.3) shrinks to be an equality (see Theorem 1), i.e.,  $h^+ = 1 - \mu_1^+$  with  $\mu_1^+$  being the first eigenvalue of the signless graph 1-Laplacian. We also extend the Courant nodal domain theorem into the signless graph 1-Laplacian and such extension (see Theorem 2) needs a slight modification of the definition of nodal domains (See Definition 2). We further modify the classical Lovász extension to the set-pair analog (see Theorem 3) which fits for the dual Cheeger problem (1.2) and with this modification, we are able to recover Theorem 1. Such generalization is necessary and essential since the usual Lovász extension can only deal with (submodular) set functions but the dual Cheeger problem involves the set-pair functions. Moreover, with the help of subdifferential techniques in nonlinear analysis, we prove that the functional of the signless graph 1-Laplacian is locally linear on any given direction. This local linearity directly implies the inverse power (IP) method [16] and thus provides an efficient implementation of the RSC algorithm, which has not been previously reported to the best of our knowledge. In a word, the main purpose of this work is twofold. One is to demonstrate the use of the signless 1-Laplacian based spectral theory into the dual Cheeger problem and the maxcut problem, while the other is to characterize the structure of the eigenvector set of the signless 1-Laplacian from which more precise insights are revealed.

This paper is organized as follows. Basic properties of the spectral theory of the signless graph 1-Laplacian are presented in Section 2. Section 3 extends the Courant nodal domain theorem into the signless graph 1-Laplacian. Section 4 provides an improved version of the Lovász extension which can be applied to the dual Cheeger problem (1.2). Section 5 analyzes the functional for the signless graph 1-Laplacian as well as the IP method in searching the minimizer, which is applied into the maxcut problem in Section 6.

## 2 Spectrum of the signless 1-Laplacian

In this section, the signless 1-Laplacian is first introduced and the corresponding nonlinear spectral theory is then presented in the spirit of [5, 7]. All these results hold for graphs with positive weight.

Let  $G = (V, E)$  be an unweighted and undirected graph with vertex set  $V = \{1, 2, \dots, n\}$  and edge set  $E$ , and  $d_i$  the degree of the vertex  $i$ .

Before going to the definition of the signless 1-Laplacian, we shall recall that for the Laplace matrix and for the signless Laplace matrix respectively. As for the Laplace matrix, first we assign an orientation on  $G$ , and then for each edge  $e = \{i, j\} \in E$ , among  $i$  and  $j$ , there must be one at the head of  $e$ , which we write  $e_h$  and the other at the tail, which we write  $e_t$ . The incidence matrix is defined by

$$B = (b_{ei}), e \in E, i \in V,$$

where

$$b_{ei} = \begin{cases} 1, & \text{if } i = e_h, \\ -1, & \text{if } i = e_t, \\ 0, & \text{if } i \notin e. \end{cases}$$

The Laplace matrix is

$$L = B^T B.$$

It is easy to see that  $L$  is independent to the orientation.

As to the signless Laplace matrix, we don't need to assign an orientation on  $G$ , but define the incidence matrix directly:

$$B^+ = (b_{ei}^+), \quad e \in E, \quad i \in V,$$

where

$$b_{ei}^+ = \begin{cases} 1, & \text{if } i \in e, \\ 0, & \text{if } i \notin e. \end{cases}$$

The signless Laplace matrix is then defined by

$$Q = (B^+)^T (B^+).$$

Obviously,  $Q = 2D - L$  with  $D = \text{diag}(d_1, \dots, d_n)$ .

While in the field of partial differential equation, the Laplacian is

$$\Delta u = \text{div}(\text{grad } u) := \nabla^T \nabla u,$$

and the 1-Laplacian is formally introduced

$$\Delta_1 u = \text{div}\left(\frac{\nabla u}{|\nabla u|}\right) := \nabla^T \left(\frac{\nabla u}{|\nabla u|}\right),$$

where  $\nabla = \text{grad}$  (resp.  $\nabla^T = \text{div}$ ) denotes the gradient (resp. divergence). In a simple analogy, we can naturally extend the graph Laplacian  $L$  to

$$\Delta_1 \mathbf{x} = B^T \left(\frac{B\mathbf{x}}{|B\mathbf{x}|}\right) := B^T \text{Sgn}(B\mathbf{x})$$

on graphs, i.e., the graph 1-Laplacian, where

$$\text{Sgn}(t) = \begin{cases} 1, & \text{if } t > 0, \\ -1, & \text{if } t < 0, \\ [-1, 1], & \text{if } t = 0. \end{cases}$$

The nonlinear spectral theory of the graph 1-Laplacian has been well documented [5, 17] as well as shows several interesting applications [4, 6, 7].

Returning to the signless graph 1-Laplacian, from the signless graph Laplacian  $Q$ , we similarly define

$$\Delta_1^+ \mathbf{x} = (B^+)^T \left(\frac{B^+ \mathbf{x}}{|B^+ \mathbf{x}|}\right) := (B^+)^T \text{Sgn}(B^+ \mathbf{x}),$$

the coordinate form of which reads

$$(\Delta_1^+ \mathbf{x})_i = \left\{ \sum_{j \sim i} z_{ij}(\mathbf{x}) \left| z_{ij}(\mathbf{x}) \in \text{Sgn}(x_i + x_j), \quad z_{ji}(\mathbf{x}) = z_{ij}(\mathbf{x}), \quad \forall j \sim i \right. \right\}, \quad i = 1, 2, \dots, n, \quad (2.1)$$

where  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ ,  $j \sim i$  denotes vertex  $j$  being adjacent to vertex  $i$ ,  $\sum_{j \sim i}$  means the summation over all vertices adjacent to vertex  $i$ .

From the variational point of view, the signless 1-Laplacian  $\Delta_1^+ \mathbf{x}$  is nothing but the subdifferential of the convex function

$$I^+(\mathbf{x}) = \sum_{i \sim j} |x_i + x_j|, \quad \mathbf{x} \in \mathbb{R}^n. \quad (2.2)$$

**Proposition 1.** For any  $\mathbf{x} \in \mathbb{R}^n$ , we have

$$\Delta_1^+ \mathbf{x} = \partial I^+(\mathbf{x}). \quad (2.3)$$

The proof is standard in convex analysis. For readers' convenience, we give the proof in Appendix A.

Let

$$X = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| = 1\}, \quad \|\mathbf{x}\| := \sum_{i=1}^n d_i |x_i|. \quad (2.4)$$

Parallel to the spectral theory of 1-Laplacian on graphs, a pair  $(\mu^+, \mathbf{x}) \in \mathbb{R}^1 \times X$  is called an eigenpair of the signless 1-Laplacian, if

$$\mathbf{0} \in \Delta_1^+ \mathbf{x} - \mu^+ D \text{Sgn}(\mathbf{x}), \quad (\text{or } \mu^+ D \text{Sgn}(\mathbf{x}) \cap \Delta_1^+ \mathbf{x} \neq \emptyset), \quad (2.5)$$

where  $\text{Sgn}(x) = (\text{Sgn}(x_1), \text{Sgn}(x_2), \dots, \text{Sgn}(x_n))^T$ . In the coordinate form, it becomes

$$\begin{cases} \text{There exist } z_{ij}(\mathbf{x}) \in \text{Sgn}(x_i + x_j) \text{ such that} \\ z_{ji}(\mathbf{x}) = z_{ij}(\mathbf{x}), \forall i \sim j \text{ and } \sum_{j \sim i} z_{ij}(\mathbf{x}) \in \mu^+ d_i \text{Sgn}(x_i), \quad i = 1, \dots, n. \end{cases} \quad (2.6)$$

In the eigenpair  $(\mu^+, \mathbf{x})$ ,  $\mathbf{x}$  is the eigenvector while  $\mu^+$  is the corresponding eigenvalue.

**Proposition 2.** If  $(\mu^+, \mathbf{x})$  is a  $\Delta_1^+$ -eigenpair, then  $I^+(\mathbf{x}) = \mu^+ \in [0, 1]$ .

*Proof.* Letting  $z_{ij} = z_{ij}(\mathbf{x})$ , it follows from Eq. (2.6) that

$$\begin{aligned} I^+(\mathbf{x}) &= \sum_{j \sim i} |x_i + x_j| = \sum_{j \sim i} (x_i + x_j) z_{ij} \\ &= \sum_{j \sim i} (z_{ij} x_i + z_{ji} x_j) = \sum_{i=1}^n \sum_{j \sim i} z_{ij} x_i \\ &= \sum_{i=1}^n \mu^+ d_i c'_i x_i = \mu^+ \sum_i d_i |x_i| = \mu^+ \|\mathbf{x}\| = \mu^+, \end{aligned}$$

where  $c'_i \in \text{Sgn } x_i$ .

At last,  $0 \leq \mu^+ = I^+(\mathbf{x}) \leq \|\mathbf{x}\| = 1$ . □

Through the Lagrange multiplier, an eigenvalue problem can usually be transformed to a critical point problem of a differentiable functional constrained on a differential manifold. Now  $I^+$  is only locally Lipschitzian, and  $X$  is only a locally Lipschitzian manifold, thanks to [5], the critical point theory has been well developed in this setting. Let  $K^+$  be the critical set of  $I^+|_X$ , and  $S^+$  be the set of all eigenvectors of the signless 1-Laplacian, then we have

**Proposition 3** (Theorem 4.11 in [5]).

$$K^+ = S^+.$$

Similar to the 1-Laplacian on graphs, the eigenvectors of the signless 1-Laplacian corresponding to the same eigenvalue also may be very abundant.

**Definition 1** (Ternary vector). A vector  $\mathbf{x}$  in  $\mathbb{R}^n \setminus \{\mathbf{0}\}$  is said to be a ternary vector if there exist two disjoint subsets  $A$  and  $B$  of  $V$  such that

$$(\mathbf{x}_{A,B})_i = \begin{cases} 1/\text{vol}(A \cup B), & \text{if } i \in A, \\ -1/\text{vol}(A \cup B), & \text{if } i \in B, \\ 0, & \text{if } i \notin A \cup B. \end{cases}$$

**Lemma 1.** If  $(\mu^+, \mathbf{x})$  is an eigenpair of  $\Delta_1^+$ , then  $\forall t \in [0, 1]$ ,  $(\mu^+, \mathbf{x}_{A,B})$  is also an eigenpair where  $\mathbf{x}_{A,B}$  is a ternary vector with  $A = \{i \in V : x_i > 0\}$  and  $B = \{i \in V : x_i < 0\}$ .

*Proof.* Let  $\hat{\mathbf{x}} = \mathbf{x}_{A,B}$ . Then, by the definition of  $\hat{\mathbf{x}}$ , there holds  $\text{Sgn}(\hat{x}_i) \supset \text{Sgn}(x_i)$ ,  $i = 1, 2, \dots, n$ . Now, we begin to verify that  $\text{Sgn}(\hat{x}_i + \hat{x}_j) \supset \text{Sgn}(x_i + x_j)$ ,  $j \sim i$ ,  $i = 1, 2, \dots, n$ .

If  $x_i x_j \geq 0$ , then  $\hat{x}_i \hat{x}_j \geq 0$  and thus it is easy to check that  $\text{Sgn}(\hat{x}_i + \hat{x}_j) = \text{Sgn}(x_i + x_j)$ . If  $x_i x_j < 0$ , then  $\hat{x}_i \hat{x}_j < 0$  and hence  $\text{Sgn}(\hat{x}_i + \hat{x}_j) = \text{Sgn}(0) = [-1, 1] \supset \text{Sgn}(x_i + x_j)$ . Therefore, we have proved that  $\text{Sgn}(\hat{x}_i + \hat{x}_j) \supset \text{Sgn}(x_i + x_j)$ ,  $j \sim i$ ,  $i = 1, 2, \dots, n$ .

Since  $(\mu^+, \mathbf{x})$  is an eigenpair, there exist  $z_{ij}(\mathbf{x}) \in \text{Sgn}(x_i + x_j)$  satisfying

$$z_{ji}(\mathbf{x}) = z_{ij}(\mathbf{x}), \forall i \sim j \text{ and } \sum_{j \sim i} z_{ij}(\mathbf{x}) \in \mu^+ d_i \text{Sgn}(x_i), \quad i = 1, \dots, n.$$

If one takes  $z_{ij}(\hat{\mathbf{x}}) = z_{ij}(\mathbf{x})$ ,  $\forall i \sim j$ , then  $z_{ij}(\hat{\mathbf{x}}) \in \text{Sgn}(\hat{x}_i + \hat{x}_j)$  and they also satisfy

$$z_{ji}(\hat{\mathbf{x}}) = z_{ij}(\hat{\mathbf{x}}), \forall i \sim j \text{ and } \sum_{j \sim i} z_{ij}(\hat{\mathbf{x}}) \in \mu^+ d_i \text{Sgn}(\hat{x}_i), \quad i = 1, \dots, n.$$

Consequently,  $(\mu^+, \hat{\mathbf{x}})$  is an eigenpair, too. This completes the proof.  $\square$

Based on all above results, we are able to build up an equality connecting the first eigenvalue  $\mu_1^+$  of the signless 1-Laplacian and the dual Cheeger constant  $h^+$ .

**Theorem 1.**

$$1 - h^+(G) = \mu_1^+ = \min_{\mathbf{x} \neq \mathbf{0}} \frac{I^+(\mathbf{x})}{\|\mathbf{x}\|}. \quad (2.7)$$

*Proof.* First we prove the second equality. Since both  $I^+(x)$  and  $\|x\|$  are positively 1-homogenous, we have

$$\frac{I^+(\mathbf{x})}{\|\mathbf{x}\|} = I^+(\mathbf{x})|_X.$$

The minimum of the function  $I^+|_X$  is obviously a critical value, an eigenvalue of  $\Delta_1^+$  by Proposition 3, and then equals to  $\mu_1^+$ .

Then we turn out to prove the first equality. It is easy to calculate that

$$1 - \frac{2|E(V_1, V_2)|}{\text{vol}(V_1 \cup V_2)} = \frac{2|E(V_1, V_1)| + 2|E(V_2, V_2)| + E(V_1 \cup V_2, (V_1 \cup V_2)^c)}{\text{vol}(V_1 \cup V_2)} = I^+(\mathbf{x}_{V_1, V_2}),$$

where  $\mathbf{x}_{V_1, V_2}$  is a ternary vector in  $X$  defined as

$$(\mathbf{x}_{V_1, V_2})_i = \begin{cases} \frac{1}{\text{vol}(V_1 \cup V_2)} & \text{if } i \in V_1, \\ -\frac{1}{\text{vol}(V_1 \cup V_2)} & \text{if } i \in V_2, \\ 0, & \text{if } i \notin V_1 \cup V_2, \end{cases} \quad \text{for } i = 1, 2, \dots, n.$$

Therefore, we have

$$\begin{aligned}
1 - h^+(G) &= 1 - \max_{V_1 \cap V_2 = \emptyset, V_1 \cup V_2 \neq \emptyset} \frac{2|E(V_1, V_2)|}{\text{vol}(V_1 \cup V_2)} \\
&= \min_{V_1 \cap V_2 = \emptyset, V_1 \cup V_2 \neq \emptyset} \left( 1 - \frac{2|E(V_1, V_2)|}{\text{vol}(V_1 \cup V_2)} \right) \\
&= \min_{V_1 \cap V_2 = \emptyset, V_1 \cup V_2 \neq \emptyset} I^+(\mathbf{x}_{V_1, V_2}) \\
&\geq \min_{\mathbf{x} \in X} I^+(\mathbf{x}).
\end{aligned}$$

On the other hand, let  $\mathbf{x}^0$  be a minimal point of  $I^+(\mathbf{x})$  on  $X$ .

Then  $\mathbf{x}^0$  must be a critical point of  $I^+|_X$  and thus a  $\Delta_1^+$  eigenvector with respect to  $\mu_1^+$ . Accordingly, by Lemma 1 we deduce that there is a ternary vector  $\hat{\mathbf{x}}^0$  which is also a  $\Delta_1^+$  eigenvector corresponding to  $\mu_1^+$ . By the construction of  $\hat{\mathbf{x}}^0$ , there exist  $V_1^0$  and  $V_2^0$  such that

$$I^+(\hat{\mathbf{x}}^0) = 1 - \frac{2|E(V_1^0, V_2^0)|}{\text{vol}(V_1^0 \cup V_2^0)},$$

and we immediately obtain that

$$\begin{aligned}
\min_{\mathbf{x} \in X} I^+(\mathbf{x}) &= I^+(\hat{\mathbf{x}}^0) \\
&\geq \min_{V_1 \cap V_2 = \emptyset, V_1 \cup V_2 \neq \emptyset} \left( 1 - \frac{2|E(V_1, V_2)|}{\text{vol}(V_1 \cup V_2)} \right) \\
&= 1 - \max_{V_1 \cap V_2 = \emptyset, V_1 \cup V_2 \neq \emptyset} \frac{2|E(V_1, V_2)|}{\text{vol}(V_1 \cup V_2)} \\
&= 1 - h^+(G).
\end{aligned}$$

The proof is completed.  $\square$

Inspired by the work of [15], we are able to give a different proof for the inequality  $\min_{\mathbf{x} \in X} I^+(\mathbf{x}) \geq 1 - h^+(G)$  with the aid of the co-area formula [18]. The details are provided in Appendix C.

**Corollary 1.** *A connected graph  $G$  is bipartite if and only if  $\mu_1^+ = 0$ .*

*Proof.* It is known [15] that a connected  $G$  is bipartite if and only if  $h^+(G) = 1$ . And by Theorem 1,  $h^+(G) = 1$  if and only if  $\mu_1^+ = 0$ . This completes the proof.  $\square$

The above proof of Corollary 1 relies heavily on Theorem 1 as well as the dual Cheeger inequality (1.3), while a direct proof is also given in Appendix B for readers' convenience.

Now we study the spectral gap of  $\Delta_1^+$ .

**Proposition 4.** *The distance between two distinct eigenvalues of  $\Delta_1^+$  is at least  $\frac{2}{n^2(n-1)^2}$ .*

*Proof.* In virtue of Lemma 1, for given different critical values  $\mu^+$  and  $\tilde{\mu}^+$ , there exists  $A_1, A_2, B_1, B_2 \subset V$  such that  $\mu^+ = 1 - \frac{2|E(A_1, A_2)|}{\text{vol}(A_1 \cup A_2)}$  and  $\tilde{\mu}^+ = 1 - \frac{2|E(B_1, B_2)|}{\text{vol}(B_1 \cup B_2)}$ . Accordingly, we obtain

$$|\mu^+ - \tilde{\mu}^+| = \left| \frac{2|E(A_1, A_2)|}{\text{vol}(A_1 \cup A_2)} - \frac{2|E(B_1, B_2)|}{\text{vol}(B_1 \cup B_2)} \right|$$

$$\begin{aligned}
&= 2 \frac{||E(A_1, A_2)| \text{vol}(B_1 \cup B_2) - |E(B_1, B_2)| \text{vol}(A_1 \cup A_2)|}{\text{vol}(A_1 \cup A_2) \text{vol}(B_1 \cup B_2)} \\
&\geq \frac{2}{\text{vol}(A_1 \cup A_2) \text{vol}(B_1 \cup B_2)} \\
&\geq \frac{2}{n^2(n-1)^2}.
\end{aligned}$$

□

As a direct consequence of Proposition 4, we conclude that the number of  $\Delta_1^+$ -eigenvalues is finite, and then all the eigenvalues can be ordered as follows:  $0 \leq \mu_1^+ < \mu_2^+ < \dots \leq 1$ .

At the end of this section, we study the construction of the set of eigenvectors associate with a given eigenvalue. For any  $\mathbf{x} \in X$ , we introduce the set

$$\Delta(\mathbf{x}) = \{\mathbf{y} \in X : \text{Sgn}(y_i) = \text{Sgn}(x_i), i = 1, 2, \dots, n\},$$

which is a simplex in  $X$ . Let

$$\pi_{\{i,j\}} = \{\mathbf{x} \in \mathbb{R}^n | x_i + x_j = 0\}$$

be a hyperplane, the complex  $X$  is divided by the family of hyperplanes

$$\{\pi_{\{i,j\}} | i \sim j\},$$

the refined complex is denoted by  $X^+$ . Accordingly, the simplex  $\Delta(\mathbf{x})$  is divided into its refinement:

$$\Delta^+(\mathbf{x}) = \{\mathbf{y} \in \Delta(\mathbf{x}) : \text{Sgn}(y_i + y_j) = \text{Sgn}(x_i + x_j), j \sim i, i = 1, 2, \dots, n\}, \quad \forall \mathbf{x} \in X.$$

As an example, Fig. 1 cartoons the complex  $X$  and the refined complex  $X^+$  for the path graph with two vertices.

As a consequence of Lemma 1, we have

**Proposition 5.** *If  $(\mu^+, \mathbf{x})$  is an eigenpair of  $\Delta_1^+$ , then  $(\mu^+, \mathbf{y})$  is also an eigenpair for any  $\mathbf{y} \in \Delta^+(\mathbf{x})$ .*

### 3 The Courant nodal domain theorem

To an eigenvector  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  of the Laplacian on a graph, the nodal domain is defined to be the maximal connected domain of the set consisting of positive (or negative) vertices, i.e.,  $\{i \in V | x_i > 0\}$  (or  $\{i \in V | x_i < 0\}$  respectively). As to the signless 1- Laplacian, we modify the definition as follows:

**Definition 2.** *The nodal domains of an eigenvector  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  of  $\Delta_1^+$  are defined to be maximal connected components of the support set  $D(\mathbf{x}) := \{i \in V : x_i \neq 0\}$ .*

The difference between these two can be seen from a connected bipartite graph. For  $\mu_1^+ = 0$ , according to the definition for  $\Delta_1$ , any eigenvector has  $n$  nodal domains, while to the definition for signless 1-Laplacian it has 1. The reason, we prefer to use the new definition for signless Laplacian in the following study instead of the old, is due to the Courant nodal domain theorem for  $\Delta_1^+$ , which connects the order and the multiplicity of an eigenvalue with the number of nodal domains of the associate eigenvector.



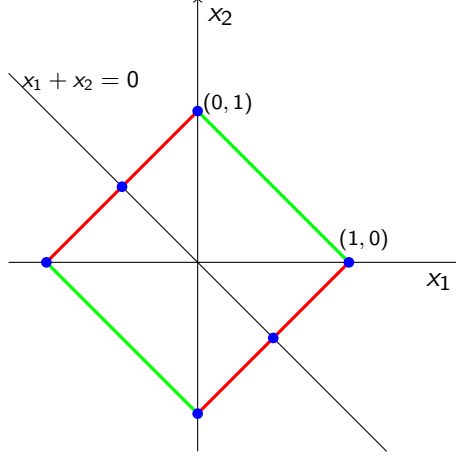


Figure 1: The complex  $X$  of the path graph with two vertices consists of four 0-cells (the four vertices of the colored square) and four 1-cells (the four sides of the colored square), while corresponding refined complex  $X^+$  consists of six 0-cells (blue) and four small 1-cells (red) and two big 1-cells (green).

**Proposition 6.** Suppose  $(\mu^+, \mathbf{x})$  is an eigenpair of the signless graph 1-Laplacian and  $D_1, \dots, D_k$  are nodal domains of  $\mathbf{x}$ . Let  $\mathbf{x}^i$  and  $\hat{\mathbf{x}}^i$  be defined as

$$x_j^i = \begin{cases} \frac{x_j}{\sum_{j \in D_i(\mathbf{x})} d_j |x_j|}, & \text{if } j \in D_i(\mathbf{x}), \\ 0, & \text{if } j \notin D_i(\mathbf{x}), \end{cases} \quad \text{and } \hat{x}_j^i = \begin{cases} \frac{1}{\sum_{j \in D_i(\mathbf{x})} d_j}, & \text{if } j \in D_i(\mathbf{x}) \text{ and } x_j > 0, \\ \frac{-1}{\sum_{j \in D_i(\mathbf{x})} d_j}, & \text{if } j \in D_i(\mathbf{x}) \text{ and } x_j < 0, \\ 0, & \text{if } j \notin D_i(\mathbf{x}), \end{cases}$$

for  $j = 1, 2, \dots, n$  and  $i = 1, 2, \dots, k$ . Then both  $(\mu^+, \mathbf{x}^i)$  and  $(\mu^+, \hat{\mathbf{x}}^i)$  are eigenpairs, too.

*Proof.* It can be directly verified that  $\text{Sgn}(\hat{x}_j^i) \supset \text{Sgn}(x_j^i) \supset \text{Sgn}(x_j)$  and  $\text{Sgn}(\hat{x}_{j'}^i + \hat{x}_j^i) \supset \text{Sgn}(x_{j'}^i + x_j^i) \supset \text{Sgn}(x_{j'} + x_j)$ ,  $j' \sim j$ ,  $j = 1, 2, \dots, n$ ,  $i = 1, 2, \dots, k$ . Then, following the proof of Lemma 1, we complete the proof.  $\square$

Parallel to the study of  $\Delta_1$ , we apply the Liusternik-Schnirelmann theory to  $\Delta_1^+$ . Now,  $I^+(\mathbf{x})$  is even, and  $X$  is symmetric, let  $T \subset X$  be a symmetric set, i.e.  $-T = T$ . The integer valued function, which is called the Krasnoselski genus of  $T$  [19, 20],  $\gamma : T \rightarrow \mathbb{Z}^+$  is defined to be:

$$\gamma(T) = \begin{cases} 0, & \text{if } T = \emptyset, \\ \min\{k \in \mathbb{Z}^+ : \exists \text{ odd continuous } h : T \rightarrow S^{k-1}\}, & \text{otherwise.} \end{cases}$$

Obviously, the genus is a topological invariant. Let us define

$$c_k = \inf_{\gamma(T) \geq k} \max_{\mathbf{x} \in T \subset X} I^+(\mathbf{x}), \quad k = 1, 2, \dots, n. \quad (3.1)$$

By the same way as already used in [5], it can be proved that these  $c_k$  are critical values of  $I^+(\mathbf{x})$ . One has

$$c_1 \leq c_2 \leq \dots \leq c_n,$$

and if  $0 \leq \dots \leq c_{k-1} < c_k = \dots = c_{k+r-1} < c_{k+r} \leq \dots \leq 1$ , the multiplicity of  $c_k$  is defined to be  $r$ . The Courant nodal domain theorem for the signless 1-Laplacian reads

**Theorem 2.** Let  $\mathbf{x}^k$  be an eigenvector with eigenvalue  $c_k$  and multiplicity  $r$ , and let  $S(\mathbf{x}^k)$  be the number of nodal domains of  $\mathbf{x}^k$ . Then we have

$$1 \leq S(\mathbf{x}^k) \leq k + r - 1.$$

*Proof.* Suppose the contrary, that there exists  $\mathbf{x}^k = (x_1, x_2, \dots, x_n)$  such that  $S(\mathbf{x}^k) \geq k + r$ . Let  $D_1(\mathbf{x}^k), \dots, D_{k+r}(\mathbf{x}^k)$  be the nodal domains of  $\mathbf{x}^k$ . Let  $\mathbf{y}^i = (y_1^i, y_2^i, \dots, y_n^i)$ , where

$$y_j^i = \begin{cases} \frac{1}{\sum_{j \in D_i(\mathbf{x}^k)} d_j}, & \text{if } j \in D_i(\mathbf{x}^k) \text{ and } x_j > 0, \\ \frac{-1}{\sum_{j \in D_i(\mathbf{x}^k)} d_j}, & \text{if } j \in D_i(\mathbf{x}^k) \text{ and } x_j < 0, \\ 0, & \text{if } j \notin D_i(\mathbf{x}^k) \text{ or } x_j = 0, \end{cases}$$

for  $i = 1, 2, \dots, k + r, j = 1, 2, \dots, n$ . By the construction, we have:

- (1) The nodal domain of  $\mathbf{y}^i$  is the  $i$ -th nodal domain of  $\mathbf{x}^k$ , i.e.,  $D(\mathbf{y}^i) = D_i(\mathbf{x}^k)$ ,
- (2)  $D(\mathbf{y}^i) \cap D(\mathbf{y}^j) = \emptyset$ ,
- (3) By Proposition 6,  $\mathbf{y}^1, \dots, \mathbf{y}^{k+r}$  are all ternary eigenvectors with the same eigenvalue

$c_k$ .

Now  $\forall \mathbf{x} = \sum_{i=1}^{k+r} a_i \mathbf{y}^i \in X, \forall v \in V, \exists$  unique  $j$  such that  $x_v = a_j y_v^j$ . Hence,  $|x_v| = \sum_{j=1}^{k+r} |a_j| |y_v^j|$ . Since  $\mathbf{x} \in X, \mathbf{y}^j \in X, j = 1, \dots, k + r$ , we have

$$1 = \sum_{v \in V} d_v |x_v| = \sum_{v \in V} d_v \sum_{j=1}^{k+r} |a_j| |y_v^j| = \sum_{j=1}^{k+r} |a_j| \sum_{v \in V} d_v |y_v^j| = \sum_{j=1}^{k+r} |a_j|.$$

Therefore, for any  $\mathbf{x} \in \text{span}(\mathbf{y}^1, \dots, \mathbf{y}^{k+r}) \cap X$ , we have

$$\begin{aligned} I^+(\mathbf{x}) &= \sum_{u \sim v} |x_u + x_v| \\ &\leq \sum_{u \sim v} \sum_{i=1}^{k+r} |a_i| |y_u^i + y_v^i| \\ &= \sum_{i=1}^{k+r} |a_i| \sum_{u \sim v} |y_u^i + y_v^i| \\ &= \sum_{i=1}^{k+r} |a_i| I^+(\mathbf{y}^i) \leq \max_{i=1,2,\dots,k+r} I^+(\mathbf{y}^i). \end{aligned}$$

Note that  $\mathbf{y}^1, \dots, \mathbf{y}^{k+r}$  are non-zero orthogonal vectors, so  $\text{span}(\mathbf{y}^1, \dots, \mathbf{y}^{k+r})$  is a  $k + r$  dimensional linear space. It follows that  $\text{span}(\mathbf{y}^1, \dots, \mathbf{y}^{k+r}) \cap X$  is a symmetric set which is homeomorphous to  $S^{k+r-1}$ . Obviously,  $\gamma(\text{span}(\mathbf{y}^1, \dots, \mathbf{y}^{k+r}) \cap X) = k + r$ . Therefore, we derive that

$$\begin{aligned} c_{k+r} &= \inf_{\gamma(A) \geq k+r} \sup_{\mathbf{x} \in A} I^+(\mathbf{x}) \\ &\leq \sup_{\mathbf{x} \in \text{span}(\mathbf{y}^1, \dots, \mathbf{y}^{k+r}) \cap X} I^+(\mathbf{x}) \end{aligned}$$

$$\begin{aligned}
&= \max_{i=1, \dots, k+r} I^+(\mathbf{y}^i) \\
&= c_k,
\end{aligned}$$

It contradicts with  $c_k < c_{k+r}$ . □

## 4 Set-pair analog of the Lovász extension

As a basic tool in combinatorial optimization theory, the Lovász extension aims at the establishment of the equivalence between discrete combination optimization and continuous function optimization. In this regard, Theorem 1 indicates that the functional  $I^+(\mathbf{x})$  could well be some kind of ‘Lovász extension’ of the dual Cheeger problem. However,  $|E(V_1, V_2)|$  and  $\text{vol}(V_1 \cup V_2)$  appeared in Eq. (1.2) cannot be a submodular set function which implies that the original Lovász extension cannot be directly applied into the dual Cheeger problem (1.2). To this end, a set-pair extension of the Lovász extension is introduced in this section.

**Definition 3** (Set-pair Lovász extension). *Let  $\mathcal{P}_2(V) = \{(A, B) : A, B \subset V \text{ with } A \cap B = \emptyset \neq A \cup B\}$ . Given  $f : \mathcal{P}_2(V) \rightarrow [0, +\infty)$ , the Lovász extension of  $f$  is a function  $f^L : \mathbb{R}^n \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}$  defined by*

$$f^L(\mathbf{x}) = \int_0^\infty f(V_t^+(\mathbf{x}), V_t^-(\mathbf{x})) dt, \quad (4.1)$$

where  $V_t^\pm(\mathbf{x}) = \{i \in V : \pm x_i > t\}$ .

It can be readily verified that  $f^L(\mathbf{x})$  is positively one-homogeneous.

**Remark 1.** We note that Eq. (4.1) can also be written as

$$f^L(\mathbf{x}) = \sum_{i=0}^{n-1} (|x_{\sigma(i+1)}| - |x_{\sigma(i)}|) f(V_i^+, V_i^-),$$

where  $\sigma : V \cup \{0\} \rightarrow V \cup \{0\}$  is a bijection related to  $\mathbf{x}$  such that

$$0 = |x_{\sigma(0)}| \leq |x_{\sigma(1)}| \leq |x_{\sigma(2)}| \leq \dots \leq |x_{\sigma(n)}|,$$

and

$$V_i^\pm := \{j \in V : \pm x_{\sigma(j)} > |x_{\sigma(i)}|\}, \quad i = 0, 1, \dots, n-1.$$

For  $A \subset V = \{1, 2, \dots, n\}$ , we use  $\mathbf{1}_A$  to denote the characteristic function of  $A$  and let  $\mathbf{1}_{A,B} = \mathbf{1}_A - \mathbf{1}_B$  for any  $A, B \subset V$  with  $A \cap B = \emptyset$ . By Definition 3, we have  $f(A, B) = f^L(\mathbf{1}_{A,B})$ .

**Theorem 3.** *Assume that  $f, g : \mathcal{P}_2(V) \rightarrow [0, +\infty)$  are two set functions with  $g(A, B) > 0$  whenever  $A \cup B \neq \emptyset$ , then there holds*

$$\min_{(A,B) \in \mathcal{P}_2(V)} \frac{f(A, B)}{g(A, B)} = \min_{\mathbf{x} \neq \mathbf{0}} \frac{f^L(\mathbf{x})}{g^L(\mathbf{x})},$$

*Proof.* On one hand,

$$\min_{(A,B) \in \mathcal{P}_2(V)} \frac{f(A,B)}{g(A,B)} = \min_{(A,B) \in \mathcal{P}_2(V)} \frac{f^L(\mathbf{1}_{A,B})}{g^L(\mathbf{1}_{A,B})} \geq \inf_{\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}} \frac{f^L(\mathbf{x})}{g^L(\mathbf{x})}.$$

On the other hand, assume  $(A_0, B_0) \in \mathcal{P}_2(V)$  being the minimizer of  $\frac{f(A,B)}{g(A,B)}$ , then  $\forall \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ , we have

$$\frac{f(A_0, B_0)}{g(A_0, B_0)} \leq \frac{f(V_t^+(\mathbf{x}), V_t^-(\mathbf{x}))}{g(V_t^+(\mathbf{x}), V_t^-(\mathbf{x}))}, \quad \forall t \geq 0.$$

where  $V_t^\pm(\mathbf{x}) = \{i \in V : \pm x_i > t\}$ . Then

$$f(V_t^+(\mathbf{x}), V_t^-(\mathbf{x})) \geq \frac{f(A_0, B_0)}{g(A_0, B_0)} g(V_t^+(\mathbf{x}), V_t^-(\mathbf{x})), \quad \forall t \geq 0.$$

Therefore,

$$\int_0^\infty f(V_t^+(\mathbf{x}), V_t^-(\mathbf{x})) dt \geq \frac{f(A_0, B_0)}{g(A_0, B_0)} \int_0^\infty g(V_t^+(\mathbf{x}), V_t^-(\mathbf{x})) dt,$$

i.e.,  $f^L(\mathbf{x}) \geq g^L(\mathbf{x}) \frac{f(A_0, B_0)}{g(A_0, B_0)}$ . This implies that

$$\min_{(A,B) \in \mathcal{P}_2(V)} \frac{f(A,B)}{g(A,B)} \leq \inf_{\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}} \frac{f^L(\mathbf{x})}{g^L(\mathbf{x})}.$$

□

In the following, we compute the Lovász extensions of the set functions  $f(A, B) = \text{vol}(A) + \text{vol}(B) - 2|E(A, B)|$  and  $g(A, B) = \text{vol}(A) + \text{vol}(B)$ , and obtain  $f^L(\mathbf{x}) = \sum_{i \sim j} |x_i + x_j|$  and  $g^L(\mathbf{x}) = \sum_{i=1}^n d_i |x_i|$ , respectively. According to Theorem 3, it follows

$$\min_{V_1 \cup V_2 \neq \emptyset = V_1 \cap V_2} \frac{\text{vol}(V_1) + \text{vol}(V_2) - 2|E(V_1, V_2)|}{\text{vol}(V_1) + \text{vol}(V_2)} = \min_{\mathbf{x} \neq \mathbf{0}} \frac{\sum_{i \sim j} |x_i + x_j|}{\sum_{i=1}^n d_i |x_i|},$$

which gives another proof of Theorem 1.

For  $g(A, B) = \text{vol}(A) + \text{vol}(B)$ , we have

$$\begin{aligned} g^L(\mathbf{x}) &= \int_0^\infty \text{vol}(V_t^+(\mathbf{x}) \cup V_t^-(\mathbf{x})) dt \\ &= \int_0^\infty \sum_i d_i \chi_{(0, |x_i|)}(t) dt \\ &= \sum_i d_i \int_0^\infty \chi_{(0, |x_i|)}(t) dt = \sum_i d_i |x_i|. \end{aligned}$$

Let  $h(A, B) = 2|E(A, B)|$ , then  $f(A, B) = g(A, B) - h(A, B)$ .

$$\begin{aligned} h^L(\mathbf{x}) &= \int_0^\infty 2|E(V_t^+(\mathbf{x}), V_t^-(\mathbf{x}))| dt \\ &= \int_0^\infty 2 \sum_{i \sim j, x_i > 0 > x_j} \chi_{(0, x_i)}(t) \chi_{(0, -x_j)}(t) dt \end{aligned}$$

$$\begin{aligned}
&= \int_0^\infty 2 \sum_{i \sim j, x_i > 0 > x_j} \chi_{(0, \min\{|x_i|, |x_j|\})}(t) dt \\
&= \sum_{i \sim j, x_i > 0 > x_j} 2 \int_0^\infty \chi_{(0, \min\{|x_i|, |x_j|\})}(t) dt \\
&= \sum_{i \sim j, x_i > 0 > x_j} 2 \min\{|x_i|, |x_j|\} \\
&= \sum_{i \sim j, x_i > 0 > x_j} (|x_i| + |x_j| - |x_i + x_j|) \\
&= \sum_{i \sim j} (|x_i| + |x_j| - |x_i + x_j|) \\
&= \sum_{i=1}^n d_i |x_i| - I^+(\mathbf{x}).
\end{aligned}$$

Therefore,

$$f^L(\mathbf{x}) = I^+(\mathbf{x}).$$

## 5 Local analysis and the inverse power method

From previous sections, we find that  $I^+(\mathbf{x})$  plays a central role in the study of the spectral theory of  $\Delta_1^+$ . Before going to the numerical study of the spectrum, we shall analyze some local properties of  $I^+(\mathbf{x})$ , by which the IP method, originally designed for the linear eigenvalue problem and recently extended to the Cheeger cut problem [16], can be applied.

The following elementary fact is well known. Let  $g : [-1, 1] \rightarrow \mathbb{R}^1$  be a convex function, then  $\partial g(0) = [g'(0, -), g'(0, +)]$ , or  $[g'(0, +), g'(0, -)]$ , where

$$g'(0, \pm) = \lim_{t \rightarrow \pm 0} \frac{g(t) - g(0)}{t}. \quad (5.1)$$

**Lemma 2.** *Let  $g : [-1, 1] \rightarrow \mathbb{R}^1$  be a convex piecewise linear function. Then*

$$g(t) = g(0) + t \max_{s \in \partial g(0)} s$$

for small  $t > 0$ .

*Proof.* On one hand, by the definition of the sub-differential, we have

$$g(t) \geq g(0) + rt, \quad \forall r \in \partial g(0),$$

it follows

$$g(t) \geq g(0) + t \max_{s \in \partial g(0)} s.$$

On the other hand, if  $g$  is piecewise linear, then  $\exists r \in \mathbb{R}^1$  such that  $g(t) = g(0) + rt$  for  $t > 0$  small, we have

$$g'(0, +) = r,$$

and then  $r = \max_{s \in \partial g(0)} s$ . The proof is completed.  $\square$

**Theorem 4.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$  be a convex piecewise linear function. Then  $\forall \mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$

$$f(\mathbf{a} + t\mathbf{v}) = f(\mathbf{a}) + t \max_{\mathbf{p} \in \partial f(\mathbf{a})} (\mathbf{v}, \mathbf{p}).$$

For a proof, we only need to define

$$g_{\mathbf{v}}(t) = f(\mathbf{a} + t\mathbf{v}), \quad \forall \mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\}.$$

Applying the above theorem to the functions  $I^+(x)$  and  $\|x\|$ , we obtain:

**Corollary 2.** Given  $\mathbf{a}, \mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ , we have

$$I^+(\mathbf{a} + t\mathbf{v}) - I^+(\mathbf{a}) = t \cdot \max_{\mathbf{p} \in \partial I^+(\mathbf{a})} (\mathbf{p}, \mathbf{v}),$$

and

$$\|\mathbf{a} + t\mathbf{v}\| - \|\mathbf{a}\| = t \cdot \max_{\mathbf{p} \in \partial \|\mathbf{a}\|} (\mathbf{p}, \mathbf{v})$$

for small  $t > 0$ .

**Theorem 5.** Assume  $\mathbf{a} \in X$  and  $\mathbf{q} \in \partial \|\mathbf{a}\|$  are fixed, then there holds: (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3), where the statements (1), (2) and (3) are claimed as follows.

(1)  $\mathbf{a}$  is not an eigenvector of the signless 1-Laplacian.

(2)

$$\min_{\mathbf{x} \in B(\mathbf{a}, \delta)} (I^+(\mathbf{x}) - \lambda(\mathbf{q}, \mathbf{x})) < 0,$$

where  $\lambda = I^+(\mathbf{a})$  and  $\delta$  can be taken as  $\min(\{|a_i + a_j|, |a_i| : i, j = 1, 2, \dots, n\} \setminus \{0\})$ .

(3)

$$I^+(\mathbf{x}_0 / \|\mathbf{x}_0\|) < I^+(\mathbf{a}),$$

where

$$\mathbf{x}_0 = \arg \min_{\mathbf{x} \in B(\mathbf{a}, \delta)} (I^+(\mathbf{x}) - \lambda(\mathbf{q}, \mathbf{x})). \quad (5.2)$$

*Proof.* (1)  $\Rightarrow$  (2):

Let  $\mathbf{q} \in \partial \|\mathbf{a}\|$  be fixed. We suppose the contrary, that  $\mathbf{a}$  is not an eigenvector of the signless 1-Laplacian, but  $\min_{\mathbf{x} \in B(\mathbf{a}, \delta)} (I^+(\mathbf{x}) - \lambda(\mathbf{q}, \mathbf{x})) \geq 0$ . Note that

$$\min_{\mathbf{x} \in B(\mathbf{a}, \delta)} (I^+(\mathbf{x}) - \lambda(\mathbf{q}, \mathbf{x})) \leq I^+(\mathbf{a}) - \lambda(\mathbf{q}, \mathbf{a}) = 0,$$

that is,  $\mathbf{a}$  is a local minimizer of the function  $I^+(\mathbf{x}) - \lambda(\mathbf{q}, \mathbf{x})$  in  $B(\mathbf{a}, \delta)$ . Hence,  $\mathbf{a}$  is a critical point of  $I^+(\mathbf{x}) - \lambda(\mathbf{q}, \mathbf{x})$ , and then

$$\mathbf{0} \in \partial(I^+(\mathbf{x}) - \lambda(\mathbf{q}, \mathbf{x}))|_{\mathbf{x}=\mathbf{a}} = \partial I^+(\mathbf{a}) - \lambda \mathbf{q}.$$

Therefore there exist  $z_{ij} \in \text{Sgn}(a_i + a_j)$  and  $q_i \in \text{Sgn}(a_i)$  such that  $\sum_{j \sim i} z_{ij} = \lambda d_i q_i$ ,  $i = 1, 2, \dots, n$ , which means that  $\mathbf{a}$  is an eigenvector of signless 1-Laplacian and  $\lambda$  is the corresponding eigenvalue. This is a contradiction.

---

**Algorithm 1:** The inverse power (IP) method.

---

**Input** :  $\mathbf{x}^0 \in X$  and  $\lambda^0 = I^+(\mathbf{x}^0)$ .

**Output:** the eigenvalue  $\lambda^{k+1}$  and the eigenvector  $\mathbf{x}^{k+1}$ .

```

1 Set  $k = 0$ ;
2 while  $\frac{|\lambda^{k+1} - \lambda^k|}{|\lambda^k|} \geq \varepsilon$  do
3   Choose  $\mathbf{v}^k \in \partial \|\mathbf{x}^k\|$ ;
4   Solve  $\mathbf{x}^{k+1} = \arg \min_{\|\mathbf{x}\|_2 \leq 1} I^+(\mathbf{x}) - \frac{I^+(\mathbf{x}^k)}{\|\mathbf{x}^k\|}(\mathbf{v}^k, \mathbf{x})$ ;
5   Set  $\mathbf{x}^{k+1} = \frac{\mathbf{x}^{k+1}}{\|\mathbf{x}^{k+1}\|}$ ;
6   Set  $\lambda^{k+1} = I^+(\mathbf{x}^{k+1})$ ;
7   Set  $k \leftarrow k + 1$ ;
8 end while

```

---

(2)  $\Rightarrow$  (3):

By the assumption, we have

$$(I^+(\mathbf{x}_0) - \lambda(\mathbf{q}, \mathbf{x}_0)) \leq \min_{\mathbf{x} \in B(\mathbf{a}, \delta)} (I^+(\mathbf{x}) - \lambda(\mathbf{q}, \mathbf{x})) < 0.$$

Since

$$(\mathbf{q}, \mathbf{x}_0) \in \sum_{i=1}^n d_i \text{Sgn}(a_i)(\mathbf{x}_0)_i \leq \sum_{i=1}^n d_i |(\mathbf{x}_0)_i| = \|\mathbf{x}_0\|,$$

it follows

$$I^+(\mathbf{x}_0) - \lambda \|\mathbf{x}_0\| < 0,$$

i.e.,  $I^+(\mathbf{x}_0/\|\mathbf{x}_0\|) < I^+(\mathbf{a})$ . □

Obviously, Eq. (5.2) in Theorem 5 indicates directly the IP method, the skeleton of which is shown in Algorithm 1. The local convergence is proved in Theorem 6.

**Theorem 6.** *The sequence  $\{I^+(\mathbf{x}^k)\}$  produced by Algorithm 1 is decreasing and convergent to an eigenvalue. Furthermore, the sequence  $\{\mathbf{x}^k\}$  produced by Algorithm 1 converges to an eigenvector of the signless graph 1-Laplacian with eigenvalue  $\lambda^* \in [1 - h^+(G), I^+(\mathbf{x}^0)]$ .*

*Proof.* It can be easily shown that  $\mu_1^+ \leq I^+(\mathbf{x}^{k+1}) \leq I^+(\mathbf{x}^k)$ . So there exists  $\lambda^* \in [\mu_1^+, I^+(\mathbf{x}^0)]$  such that  $\{I^+(\mathbf{x}^k)\}$  converges to  $\lambda^*$  decreasingly. Next we prove that  $\lambda^*$  is a  $\Delta_1^+$ -eigenvalue.

Denote

$$g(\lambda, \mathbf{v}) = \min_{\|\mathbf{x}\|_2 \leq 1} I^+(\mathbf{x}) - \lambda(\mathbf{v}, \mathbf{x}).$$

It is easy to see that  $g(\lambda, \mathbf{v})$  is uniformly continuous on  $[0, 1] \times \prod_{i=1}^n [-d_i, d_i]$  since  $\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 \leq 1\}$  and  $[0, 1] \times \prod_{i=1}^n [-d_i, d_i]$  are compact.

It follows from  $\mathbf{x}^k \in X$  and  $X$  is compact that there exists  $\mathbf{x}^* \in X$  and a subsequence  $\{\mathbf{x}^{k_i}\}$  of  $\{\mathbf{x}^k\}$  such that  $\lim_{i \rightarrow +\infty} \mathbf{x}^{k_i} = \mathbf{x}^*$ . For simplicity, We may assume without loss of generality that  $\{\mathbf{x}^{k_i}\} = \{\mathbf{x}^k\}$ , that is,  $\lim_{k \rightarrow +\infty} \mathbf{x}^k = \mathbf{x}^*$ . According to the upper semi-continuity of the subdifferential,  $\forall \epsilon > 0, \exists \delta > 0$  such that  $\partial \|\mathbf{x}\| \subset (\partial \|\mathbf{x}^*\|)_\epsilon$ , the  $\epsilon$  neighborhood of the subset  $\partial \|\mathbf{x}^*\|$ ,  $\forall \mathbf{x} \in B(\mathbf{x}^*, \delta)$ . So there exists  $N > 0$  such that  $\partial \|\mathbf{x}^k\| \subset (\partial \|\mathbf{x}^*\|)_\epsilon$  whenever  $k > N$ .

Thus,  $\mathbf{v}^k \in \partial(\|\mathbf{x}^*\|)_\epsilon$  for any  $k > N$ , which means that there is a convergent subsequence of  $\{\mathbf{v}^k\}$  (note that  $\partial\|\mathbf{x}^*\|$  is compact), still denoted it by  $\{\mathbf{v}^k\}$  for simplicity. Then  $\mathbf{v}^k \rightarrow \mathbf{v}^*$  for some  $\mathbf{v}^* \in \partial\|\mathbf{x}^*\|$ . Hence, according to the continuity of  $g$ , we have

$$g(\lambda^*, \mathbf{v}^*) = \lim_{k \rightarrow +\infty} g(\lambda^k, \mathbf{v}^k).$$

Note that

$$I^+(\mathbf{x}^*) = \lim_{k \rightarrow +\infty} I^+(\mathbf{x}^k) = \lim_{k \rightarrow +\infty} \lambda^k = \lambda^*.$$

Suppose the contrary, that  $\lambda^*$  is not an eigenvalue, then  $\mathbf{x}^*$  is not an eigenvector and so by Theorem 5, we have

$$g(\lambda^*, \mathbf{v}^*) = \min_{\|\mathbf{x}\|_2 \leq 1} I^+(\mathbf{x}) - \lambda(\mathbf{v}, \mathbf{x}) < 0,$$

which implies that  $g(\lambda^k, \mathbf{v}^k) < -\epsilon^*$  for sufficiently large  $k$  and some  $\epsilon^* > 0$ . Therefore

$$I^+(\mathbf{x}^{k+1}) - \lambda^k \|\mathbf{x}^{k+1}\| \leq I^+(\mathbf{x}^{k+1}) - \lambda^k(\mathbf{v}^k, \mathbf{x}^{k+1}) = g(\lambda^k, \mathbf{v}^k) < -\epsilon^*,$$

so

$$\lambda^{k+1} - \lambda^k = \frac{I^+(\mathbf{x}^{k+1})}{\|\mathbf{x}^{k+1}\|} - \lambda^k < -\epsilon^*/M,$$

which follows that  $\lim_{k \rightarrow +\infty} \lambda^k = -\infty$ , where  $M$  is a given positive constant satisfying  $\|\mathbf{x}\|/M \leq \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|$ . This is a contradiction and then we have finished the proof.  $\square$

The feasibility of the IP method reflects intrinsically the local linearity of  $I^+(\mathbf{x})$  on a given direction, though it was regarded as a kind of relaxation before [7]. Note in passing that the above local analysis is also applicable for the graph 1-Laplacian of the Cheeger cut problem.

## 6 Application in the maxcut problem

The minimization problem (2.7) for the dual Cheeger constant was adopted in recursively finding a partition of the vertices which maximizes the weight of edges whose endpoints are on different sides of the partition, namely a solution to the max-cut problem [14]. However, due to the lack of efficient algorithms for the dual Cheeger problem (2.7), no detailed numerical study on the performance of the signless 1-Laplacian based RSC ( $\Delta_1$ -RSC) algorithm exists in the literature. To this end, we will utilize the proposed IP method to implement the  $\Delta_1$ -RSC algorithm, and the implementation will be tested on the graphs with positive weight in G-set, a standard testbed for algorithms for the maxcut problem. It should be noted that, alternatively, the eigenvector corresponding to the largest eigenvalue of the 2-Laplacian (i.e., the normalized graph Laplacian), a continuous relaxation of the 1-Laplacian eigenvalue problem, can be also equipped into the RSC algorithm and we denote the resulting algorithm by  $\Delta_2$ -RSC.

Proposition 6 ensures theoretically the existence of ternary eigenvectors from which two disjoint subsets  $A$  and  $B$  of the vertex set are readily obtained. However, which ternary eigenvector should be chosen and how to numerically determine it play a key role in practice. A 2-thresholds spectral cut algorithm is proposed in [14] and the resulting pair  $(A, B)$  maximizes the objective function  $\frac{C}{M}$  (denoted by **ObjFun1**) where  $M$  is the weight of edges incident on  $A \cup B$  and  $C$  the weight of edges that cross  $A$  and  $B$ . It was also pointed out that another



objective function  $\frac{C+\frac{1}{2}X}{M}$  (denoted by **ObjFun2**) gives better results [21], where  $X$  is the weight of edges that have one endpoint in  $A \cup B$  and the other in the complement. We implement both and employ the MOSEK solver with CVX [22], a package for specifying and solving convex programs, to solve the convex optimization problem. Table 1 presents the numerical solutions, from which it is easily observed that: (a) With the same RSC solver, **ObjFun2** performs better than **ObjFun1**; (b) with the same object function,  $\Delta_1$ -RSC gives the cut value greater than or equal to that by  $\Delta_2$ -RSC. Overall, the  $\Delta_1$ -RSC algorithm equipped with the object function **ObjFun2** provides the best maxcut among those four solvers.

The resulting RSC algorithm has been proved to be the first algorithm for the maxcut problem with a numerical solution the cost of which is strictly larger than  $1/2$  that is not based on semidefinite programming. A simple greedy procedure cuts at least half of the total weight of edges. Here the cost of a solution refers to the ratio between the total weight of cut edges and the total weight of all edges. From Table 1, we can also easily verify such fact. Furthermore, a rigorous lower bound for the cost is determined in Theorem 7.

**Theorem 7.** *If the  $\Delta_1$ -RSC algorithm receives in input a graph whose optimum is  $1 - \epsilon$ , then it finds a solution that cuts at least a  $1 - \epsilon + \epsilon \ln 2\epsilon$  fraction of edges.*

*Proof.* By the greedy method, every graph's optimum is not less than  $\frac{1}{2}$ . So, for given graph  $G = (V, E)$ , we can assume that its optimum is  $1 - \epsilon$  for some  $\epsilon \in [0, \frac{1}{2}]$ . That is, there exists a partition  $(A, B)$  such that  $\frac{|E(A, B)|}{|E|} \geq 1 - \epsilon$  and then  $\frac{|E(A)| + |E(B)|}{|E|} \leq \epsilon$ .

Consider the  $t$ -th iteration of the algorithm, and let  $G_t = (V_t, E_t)$  be the residual graph at that iteration, and let  $|E_t| := \rho_t \cdot |E|$  be the number of edges of  $G_t$ . Then we observe that the maxcut optimum in  $G_t$  is at least  $1 - \frac{\epsilon}{\rho_t}$  because

$$\frac{|E_t(A \cap V_t)| + |E_t(B \cap V_t)|}{|E_t|} \leq \frac{|E(A)| + |E(B)|}{|E_t|} = \frac{|E(A)| + |E(B)|}{\rho_t \cdot |E|} \leq \frac{\epsilon}{\rho_t}.$$

Let  $L_t, R_t$  be the partition of found by the algorithm of Theorem 1. Let  $G_{t+1}$  be the residual graph at the following step, and  $\rho_{t+1} \cdot |E|$  the number of edges of  $G_{t+1}$ . If the algorithm stops at the  $t$ -th iteration, we shall take  $G_{t+1}$  to be the empty graph; if the algorithm discards  $L_t, R_t$  and chooses a greedy cut, we shall take  $G_{t+1}$  to be empty and  $L_t, R_t$  to be the partition given by the greedy algorithm.

We know by Theorem 1 that the algorithm will cut at least a  $1 - \frac{\epsilon}{\rho_t}$  fraction of the  $|E|(\rho_t - \rho_{t+1})$  edges incident on  $V_t$ .

Indeed, we know that at least a  $\max\{\frac{1}{2}, 1 - \frac{\epsilon}{\rho_t}\}$  fraction of those edges are cut (for small value of  $\rho_t$ , it is possible that  $1 - \frac{\epsilon}{\rho_t} < \frac{1}{2}$ , but the algorithm is always guaranteed to cut at least half of the edges incident on  $V_t$ ).

(1) If  $\rho_t \geq \rho_{t+1} > 2\epsilon$ , then

$$\begin{aligned} |E| \cdot (\rho_t - \rho_{t+1}) \cdot \left(1 - \frac{\epsilon}{\rho_t}\right) &= |E| \cdot \int_{\rho_{t+1}}^{\rho_t} \left(1 - \frac{\epsilon}{r}\right) dr \\ &\geq |E| \cdot \int_{\rho_{t+1}}^{\rho_t} \left(1 - \frac{\epsilon}{r}\right) dr. \end{aligned}$$

Table 1: Numerical solutions found for problem instances with positive weight in G-set.

Graph	V	E	$\Delta_2$ -RSC		$\Delta_1$ -RSC	
			ObjFun1	ObjFun2	ObjFun1	ObjFun2
G1	800	19176	11221	11262	11221	11319
G2	800	19176	11283	11304	11283	11358
G3	800	19176	11298	11343	11298	11405
G4	800	19176	11278	11322	11278	11415
G5	800	19176	11370	11378	11370	11465
G14	800	4694	2889	2889	2892	2955
G15	800	4661	2771	2840	2784	2895
G16	800	4672	2841	2847	2845	2925
G17	800	4667	2866	2895	2883	2929
G22	2000	19990	12876	12942	12876	13094
G23	2000	19990	12817	12945	12817	13085
G24	2000	19990	12826	12884	12826	13056
G25	2000	19990	12781	12840	12781	13025
G26	2000	19990	12752	12878	12752	12994
G35	2000	11778	7194	7223	7203	7400
G36	2000	11766	7124	7218	7128	7322
G37	2000	11785	7162	7262	7166	7389
G38	2000	11779	7122	7162	7130	7388
G43	1000	9990	6395	6432	6402	6500
G44	1000	9990	6439	6439	6439	6533
G45	1000	9990	6364	6408	6364	6485
G46	1000	9990	6389	6392	6389	6491
G47	1000	9990	6353	6401	6353	6465
G48	3000	6000	6000	6000	6000	6000
G49	3000	6000	6000	6000	6000	6000
G50	3000	6000	5880	5880	5880	5880
G51	1000	5909	3645	3649	3648	3731
G52	1000	5916	3645	3680	3654	3739
G53	1000	5914	3634	3658	3638	3715
G54	1000	5916	3655	3708	3660	3720

(2) If  $\rho_t \geq 2\varepsilon \geq \rho_{t+1}$ , then

$$|E| \cdot (\rho_t - 2\varepsilon) \cdot \left(1 - \frac{\varepsilon}{\rho_t}\right) + |E| \cdot (2\varepsilon - \rho_{t+1}) \cdot \frac{1}{2} \geq |E| \cdot \int_{2\varepsilon}^{\rho_t} \left(1 - \frac{\varepsilon}{r}\right) dr + |E| \cdot \int_{\rho_{t+1}}^{2\varepsilon} \frac{1}{2} dr.$$

(3) If  $2\varepsilon > \rho_t \geq \rho_{t+1}$ , then

$$|E| \cdot (\rho_t - \rho_{t+1}) \cdot \frac{1}{2} = |E| \cdot \int_{\rho_{t+1}}^{\rho_t} \frac{1}{2} dr.$$

Summing those bounds above, we have that the number of edges cut by the algorithm is at least

$$\begin{aligned} \sum_t |E| \cdot (\rho_t - \rho_{t+1}) \cdot \max\left\{\frac{1}{2}, 1 - \frac{\varepsilon}{\rho_t}\right\} &\geq |E| \cdot \int_{2\varepsilon}^1 \left(1 - \frac{\varepsilon}{r}\right) dr + |E| \cdot \int_0^{2\varepsilon} \frac{1}{2} dr \\ &= ((1 - 2\varepsilon) - \varepsilon(\ln 1 - \ln 2\varepsilon) + \varepsilon) \cdot |E| \\ &= (1 - \varepsilon + \varepsilon \ln 2\varepsilon) \cdot |E|. \end{aligned}$$

The proof is finished.  $\square$

As for the ratio between the total weight of cut edges and the optimum, it was first proved that the approximation ratio of the  $\Delta_1$ -RSC algorithm is of at least 0.531 [14], and later improved it to 0.614 [23]. We conjecture that this lower bound cannot be improved greater than 0.768 for  $\min_{0 \leq \varepsilon \leq 1/2} \frac{1 - \varepsilon + \varepsilon \ln 2\varepsilon}{1 - \varepsilon} \geq 0.768$ .

## Acknowledgement

This research was supported by grants from the National Natural Science Foundation of China (Nos. 11371038, 11421101, 11471025, 61121002, 91330110).

## Appendix A: Proof of Proposition 1

By the properties of subgradient, we have

$$\partial I^+(\mathbf{x}) = \sum_{i \sim j} \partial |x_i + x_j|. \quad (\text{A.1})$$

Next we will prove that

$$\partial |x_i + x_j| = \{z_{ij}\mathbf{e}_i + z_{ji}\mathbf{e}_j : z_{ij} \in \text{Sgn}(x_i + x_j), z_{ji} = z_{ij}\}, \quad (\text{A.2})$$

where  $\{\mathbf{e}_i\}_{i=1}^n$  is a standard orthogonal basis of  $\mathbb{R}^n$ . In fact, one may assume without loss of generality that  $i = 1, j = 2$ . By the definition of subgradient, we only need to find the  $\mathbf{x}^* \in \mathbb{R}^n$  such that for any  $\mathbf{y} \in \mathbb{R}^n$ ,  $(\mathbf{x}^*, \mathbf{y} - \mathbf{x}) \leq |y_1 + y_2| - |x_1 + x_2|$ , that is,

$$\sum_{i=1}^n x_i^*(y_i - x_i) \leq |y_1 + y_2| - |x_1 + x_2|, \quad \text{for any } y_i \in \mathbb{R}, i = 1, 2, \dots, n. \quad (\text{A.3})$$

It follows from Eq. (A.3) that  $x_3^* = \dots = x_n^* = 0$  and then the above inequality Eq. (A.3) becomes

$$x_1^*(y_1 - x_1) + x_2^*(y_2 - x_2) \leq |y_1 + y_2| - |x_1 + x_2|. \quad (\text{A.4})$$

For any  $h \in \mathbb{R}$ , take  $y_1 = x_1 + h$ ,  $y_2 = x_2 - h$ , then Eq. (A.4) can be simplified into  $(x_1^* - x_2^*)h \leq 0$ . Hence, by the arbitrariness of  $h$ , we have  $x_1^* - x_2^* = 0$ , which implies that Eq. (A.4) is actually

$$x_1^*(y_1 + y_2 - x_1 - x_2) \leq |y_1 + y_2| - |x_1 + x_2|. \quad (\text{A.5})$$

For the case  $x_1 + x_2 = 0$ , Eq. (A.5) becomes  $x_1^*(y_1 + y_2) \leq |y_1 + y_2|$  for any  $y_1, y_2 \in \mathbb{R}$ , which is equivalent to  $x_1^* \in [-1, 1] = \text{Sgn}(0)$ .

For  $x_1 + x_2 > 0$ , plugging  $y_1 = x_1/2$  and  $y_2 = x_2/2$  into Eq. (A.5), we immediately get  $-x_1^*(x_1 + x_2)/2 \leq -|x_1 + x_2|/2$ , which implies  $x_1^* \geq 1$ . One may similarly take  $y_1 = 2x_1$  and  $y_2 = 2x_2$  into Eq. (A.5). This deduces that  $x_1^* \leq 1$ . Hence, we have  $x_1^* = 1$ . On the other hand, Eq. (A.5) clearly holds when  $x_1^* = 1$ .

Analogously, if  $x_1 + x_2 < 0$ , we have  $x_1^* = -1$ . By these discussions, one can readily arrive at  $\mathbf{x}^* = x_1^* \mathbf{e}_1 + x_2^* \mathbf{e}_2$ , where  $x_1^* = x_2^* \in \text{Sgn}(x_1 + x_2)$ .

## Appendix B: Proof of Corollary 1

If  $G$  is connected and bipartite, then there exists a partition  $(V_1, V_2)$  of  $V$  such that  $u \sim v$  implies  $u \in V_1$  and  $v \in V_2$ . Let  $\mathbf{x} : V \rightarrow \mathbb{R}$  be defined as

$$x_i = \begin{cases} \frac{1}{\text{vol}(V)}, & \text{if } i \in V_1, \\ -\frac{1}{\text{vol}(V)}, & \text{if } i \in V_2, \end{cases} \quad \text{for } i = 1, 2, \dots, n,$$

then

$$I^+(\mathbf{x}) = \sum_{u \sim v} |x_u + x_v| \leq \sum_{u \in V_1, v \in V_2} |x_u + x_v| = 0.$$

And we can take  $z_{uv} = 0$  and  $\mu^+ = 0$ .

On the other hand, assume that  $G$  is connected,  $\mu_1^+ = 0$  and  $\mathbf{x}$  is the corresponding eigenvector. Let  $V_1 = \{u : x_u > 0\}$  and  $V_2 = \{v : x_v < 0\}$ . We first prove that  $(V_1, V_2)$  form a partition of  $V$ . Suppose the contrary, that there exists  $u \in V$  such that  $x_u = 0$ , then by the connectedness of  $G$  and  $I^+(\mathbf{x}) = 0$ ,  $x_v = 0$  holds for any  $v \in V$ . This is a contradiction with  $\mathbf{x} \in X$ . Thus,  $(V_1, V_2)$  is a partition of  $V$ . At last, we prove that if  $u \sim v$ , then  $u \in V_1, v \in V_2$  or  $u \in V_2, v \in V_1$ . In fact, suppose the contrary, that there exist  $i \in \{1, 2\}$  and  $u, v \in V_i$  with  $u \sim v$ , then  $I^+(\mathbf{x}) \geq |x_u + x_v| > 0$ , which is a contradiction with  $I^+(\mathbf{x}) = 0$ . Therefore,  $(V_1, V_2)$  is a bipartite partition and  $G$  is bipartite.

## Appendix C: Another proof of Theorem 1

We only prove the inequality  $\inf_{\mathbf{x} \in X} I^+(\mathbf{x}) \geq 1 - h^+(G)$ . The key ingredient for the proof is the well-known co-area formula [18].

**Lemma 3.** For any  $\mathbf{x} \in X$  with  $x_i \geq 0$ ,  $i = 1, 2, \dots, n$ , there exists  $S \subset D(\mathbf{x})$  such that  $\frac{|\partial S|}{\text{vol}(S)} \leq I^-(\mathbf{x}) := \sum_{i \sim j} |x_i - x_j|$ .

*Proof.* For  $t \geq 0$ , we denote

$$S_t = \{i \in D(\mathbf{x}) : x_i \geq t\}.$$

Then we have

$$\int_0^\infty \text{vol}(S_t) dt = \int_0^\infty \sum_i d_i \chi_{(0, x_i]}(t) dt = \sum_i d_i \int_0^\infty \chi_{(0, x_i]}(t) dt = \sum_i d_i |x_i| = 1$$

and

$$\int_0^\infty |\partial(S_t)| dt = \int_0^\infty \sum_{i \sim j} \chi_{(x_i, x_j]}(t) dt = \sum_{i \sim j} \int_0^\infty \chi_{(x_i, x_j]}(t) dt = \sum_{i \sim j} |x_i - x_j|,$$

which imply

$$\frac{\int_0^\infty |\partial(S_t)| dt}{\int_0^\infty \text{vol}(S_t) dt} = \sum_{i \sim j} |x_i - x_j| = I^-(\mathbf{x}).$$

Thus there exists  $t > 0$  such that  $\frac{|\partial(S_t)|}{\text{vol}(S_t)} \leq I^-(\mathbf{x})$ . □

**Lemma 4.** For any  $\mathbf{x} \in X$ , there exist two subsets  $V_1, V_2 \subset D(\mathbf{x})$  such that  $V_1 \cap V_2 = \emptyset$ ,  $V_1 \cup V_2 \neq \emptyset$  and

$$I^+(\mathbf{x}) \geq 1 - \frac{2|E(V_1, V_2)|}{\text{vol}(V_1 \cup V_2)}.$$

This lemma is derived from the combination of Lemma 3 and a construction in Bauer and Jost [15].

*Proof.* Construct a new graph  $G' = (V', E')$  from the original graph  $G$  in the following way. Duplicate all the vertices in  $D^+(\mathbf{x})$  and  $D^-(\mathbf{x})$ . Denote by  $u'$  the new vertices duplicated from  $u$ . For any edge  $\{u, v\}$  such that  $u, v \in D^+(\mathbf{x})$  or  $u, v \in D^-(\mathbf{x})$ , replace it by two new edges  $\{u', v'\}, \{v', u'\}$  with the same weight. All the other vertices, edges and weights are unchanged.

Consider  $\mathbf{x}' : V' \rightarrow \mathbb{R}$ ,

$$x'_i = \begin{cases} |x_i| & i \in D(\mathbf{x}), \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } i = 1, 2, \dots, n.$$

It is easy to verify that

$$I^+(\mathbf{x}) \geq I^-(\mathbf{x}').$$

Now by Lemma 3, there exists  $S \subset D(\mathbf{x})$  such that  $\frac{|\partial S|}{\text{vol}(S)} \leq I^-(\mathbf{x}')$ . Denote  $V_1 = S \cap D^+(\mathbf{x})$ ,  $V_2 = S \cap D^-(\mathbf{x})$ . Then we have  $V_1 \cap V_2 = \emptyset$ ,  $V_1 \cup V_2 \neq \emptyset$  and

$$\begin{aligned} \frac{E'(S, S^c)}{\text{vol}(S)} &= \frac{2E(V_1, V_1) + 2E(V_2, V_2) + E(S, S^c)}{\text{vol}(V_1 \cup V_2)} \\ &= 1 - \frac{2|E(V_1, V_2)|}{\text{vol}(V_1 \cup V_2)}. \end{aligned}$$

□

By Lemma 4, for any  $\mathbf{x} \in X$ , there exist two subsets  $V_1, V_2 \subset D(\mathbf{x})$  such that  $V_1 \cap V_2 = \emptyset$ ,  $V_1 \cup V_2 \neq \emptyset$  and

$$I^+(\mathbf{x}) \geq 1 - \frac{2|E(V_1, V_2)|}{\text{vol}(V_1 \cup V_2)} \geq 1 - h^+(G).$$

Thus,

$$\inf_{\mathbf{x} \in X} I^+(\mathbf{x}) \geq 1 - h^+(G).$$

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